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J. Math. Anal. Appl. 311 (2005) 744–754

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Note

A note on the non-formation of vacuum states for compressible Navier–Stokes equations

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Received 14 November 2004

Available online 31 March 2005

Submitted by M.D. Gunzburger

Abstract

We give a refinement of Lemma 2.2 in [D. Hoff, J.A. Smoller, Non-formation of vacuum states for compressible Navier–Stokes equations, Comm. Math. Phys. 216 (2001) 255–276] and complete the proof of non-formation of vacuum states for one-dimensional compressible Navier–Stokes equation given there.

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Keywords: Compressible Navier–Stokes equations; Vacuum states

1. Introduction

It is proved in the elegant and important paper [1] that weak solutions of the Navier–Stokes equations for compressible fluid flow in one space dimension do not exhibit vacuum states provided that no vacuum states are present initially. Before stating the main purpose

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doi:10.1016/j.jmaa.2005.02.065

of our present note, we first outline the main arguments developed in [1] to deduce the desired result. Note that all the subsequent terminologies and notations are taken from [1].

The main idea in [1] is to show that the hypothesis $\rho(x,t)=0$ for some $t>0$ and for a.e. x in some open subset of $\mathbf{R}$ will lead to a contradiction. For this purpose, suppose that $\rho(x,t_1)=0$ a.e. on (a,b) where a is minimal and b is maximal, the finite average convection speed implies that there must be nearby vacuum states at nearby times. In fact Lemma 2.2 in [1] shows that

Lemma 1.1. *Let $t_1<T$ and suppose that $\rho(\cdot,t_1)=0$ a.e. on open interval (a,b). Let*

$$t_0=\inf\left\{t\in[0,t_1]:\ \int_t^{t_1}\big\|u(\cdot,s)\big\|_{L^\infty(a,b)}\,ds<\frac{1}{2}(b-a)\right\} \tag{1}$$

and

$$t_2=\sup\left\{t\in[t_1,T]:\ \int_{t_1}^{t}\big\|u(\cdot,s)\big\|_{L^\infty(a,b)}\,ds<\frac{1}{2}(b-a)\right\}. \tag{2}$$

Then $t_0<t_1<t_2$, and for any $t\in(t_0,t_2)$, $\rho(\cdot,t)=0$ a.e. on the interval

$$\left(a+\left|\int_{t_1}^{t}\big\|u(\cdot,s)\big\|_{L^\infty(a,b)}\,ds\right|,\ b-\left|\int_{t_1}^{t}\big\|u(\cdot,s)\big\|_{L^\infty(a,b)}\,ds\right|\right). \tag{3}$$

Let t_0 be as in the statement of Lemma 1.1, and define for $t\in(t_0,t_1)$,

$$\begin{cases} y(t)=\inf\big\{x:\ \rho(\cdot,t)=0 \text{ a.e. on } \big(x,\frac{a+b}{2}\big)\big\},\\ z(t)=\sup\big\{x:\ \rho(\cdot,t)=0 \text{ a.e. on } \big(\frac{a+b}{2},x\big)\big\}, \end{cases} \tag{4}$$

with $y(t_1)=a$, $z(t_1)=b$. Then it is proved in [1] that $y(t)$ and $z(t)$ are absolutely continuous and there exists a constant $\tau\geqslant 0$ such that $y(t)$ and $z(t)$ have absolutely continuous extensions to the time τ, $y(\tau)=z(\tau)$, and there is an $L>0$ such that for all $\tau\in(\tau,t_1]$, $-L\leqslant y(t)\leqslant z(t)\leqslant L$.

Set

$$V=\big\{(x,t):\ y(t)<x<z(t),\ \tau<t\leqslant t_1\big\},$$

it is shown in [1] that there exist functions $\alpha(t),\beta(t)\in L^1_{\mathrm{loc}}((\tau,t_1])$ such that $u(x,t)=\alpha(t)x+\beta(t)$ in $\mathcal{D}'(V)$ and $u(x,t)=\alpha(t)x+\beta(t)$ for all x and almost all t in V.

Now, what is the most difficult part of the analysis in [1] is to show that the integral curves of u which starts in V must remains in V on $[t_0,t_1]$. From which one can deduce that

$$\lim_{t\to\tau+}\int_t^{t_1}\alpha(s)\,ds=\infty.$$

This violating the hypothesis that the momentum remains locally finite and this contradiction implies that weak solutions of the Navier–Stokes equations for compressible fluid flow in one space dimension do not exhibit vacuum states provided that no vacuum states are present initially.

To prove that the integral curves of u which starts in V must remains in V on $[t_0, t_1]$, the key point in [1] is to deduce the following differential inequalities:

$$\begin{cases} \frac{dz(t)}{dt} \leqslant \alpha(t)z(t) + \beta(t), \\ \frac{dy(t)}{dt} \geqslant \alpha(t)y(t) + \beta(t), \end{cases} \tag{5}$$

for almost all $t \in (\tau, t_1]$. Since the proof of (5) is closely related to the main purpose of this note, we outline the proof of $(5)_1$ in the following.

As in [1], one only need to prove that $(5)_1$ holds for $\bar{t} \notin A \cup D \cup E \cup F$. Suppose not, then there exits an $\varepsilon > 0$ such that for t near $\bar{t}$ and $t > \bar{t}$,

$$z(t) \geqslant z(\bar{t}) + (t - \bar{t})(\bar{u} + \varepsilon), \quad \bar{u} = \alpha(\bar{t})\bar{z} + \beta(\bar{t}). \tag{6}$$

Because $u(\cdot, t) \in H^1_{\mathrm{loc}}$, one can find $h > 0$ such that if $|x - \bar{z}| < h$,

$$\left|u(x, \bar{t}) - \bar{u}\right| < \frac{\varepsilon}{2} \tag{7}$$

and

$$y(\bar{t}) < \bar{z} - h. \tag{8}$$

Then choose B_{jk} such that

$$\bar{z} \in B_{jk} \subset [\bar{z} - h, \bar{z} + h].$$

Let $B_{jk} = (c, d)$ and choose e such that

$$\bar{z} - h < c < e < \bar{z} < d < \bar{z} + h.$$

Since $y(t)$ and $z(t)$ are continuous functions, one can thus find $\Delta t > 0$ such that

$$|t - \bar{t}| < \Delta t \quad \Rightarrow \quad y(t) < c, \quad e \leqslant z(t) \leqslant d.$$

Thus if $|t - \bar{t}| < \Delta t$, $\rho(\cdot, t) = 0$ a.e. on $(c, e) \subset (y(t), z(t))$. Then by Lemma 1.1, $\rho(\cdot, \bar{t}) =$ 0 a.e. on

$$\left(c + \int_{\bar{t}}^{t} \left\|u(x, s)\right\|_{L^\infty(c,e)} ds,\ z(t) - \int_{\bar{t}}^{t} \left\|u(x, s)\right\|_{L^\infty(c,e)} ds\right). \tag{9}$$

Consequently

$$\bar{z} \geqslant z(t) - \int_{\bar{t}}^{t} \left\|u(x, s)\right\|_{L^\infty(c,e)} ds \geqslant z(t) - \int_{\bar{t}}^{t} \left\|u(x, s)\right\|_{L^\infty(B_{jk})} ds. \tag{10}$$

Combining (6) and (10) deduce

$$\bar{z} + \int_{\bar{t}}^{t} \left\|u(x, s)\right\|_{L^\infty(B_{jk})} ds \geqslant \bar{z} + (t - \bar{t})(\bar{u} + \varepsilon)$$

so that

$$\bar{u} + \varepsilon \leqslant \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} \left\| u(x, s) \right\|_{L^\infty(B_{jk})} ds.$$

Letting $t \to \bar{t}+$ in the above inequality, one gets

$$\bar{u} + \varepsilon \leqslant \left\| u(x, \bar{t}) \right\|_{L^\infty(B_{jk})}. \tag{11}$$

Having obtained (11), it is claimed in [1] that it contradicts (7) since $B_{jk} \subset [\bar{z}-h, \bar{z}+h]$. We note however that (11) contradicts (7) only if $\bar{u} \geqslant 0$. In such a sense, the proof in [1] is incomplete and the main purpose of our present note is to give a refinement of the estimate (3) and then, to complete the proof of [1]. Our main result in this note is to get the following estimate on the vacuum interval at nearby time.

Lemma 1.2 (Main result). *Under the notations listed in Lemma* 1.1*, we have for any* $t \in (t_1, t_2)$*,* $\rho(\cdot, t) = 0$ *a.e. on the interval*

$$\left(a + \int_{t_1}^{t} \sup_{x \in (\frac{3a-b}{2}, \frac{3b-a}{2})} u(x, s)\, ds,\ b + \int_{t_1}^{t} \inf_{x \in (\frac{3a-b}{2}, \frac{3b-a}{2})} u(x, s)\, ds \right) \tag{12}$$

and for any $t \in (t_0, t_1)$*,* $\rho(\cdot, t) = 0$ *a.e. on the interval*

$$\left(a - \int_{t}^{t_1} \inf_{x \in (\frac{3a-b}{2}, \frac{3b-a}{2})} u(x, s)\, ds,\ b - \int_{t}^{t_1} \sup_{x \in (\frac{3a-b}{2}, \frac{3b-a}{2})} u(x, s)\, ds \right). \tag{13}$$

Having obtained Lemma 1.2, we now turn to prove $(5)_1$. To this end, we choose $B_{jk} = (c, d)$ such that

$$y(\bar{t}) < \bar{z} - h < c < \bar{z} < d < \bar{z} + h$$

and

$$4\bar{z} - c < 3d. \tag{14}$$

In fact by choosing $h > 0$ sufficiently small such that $h < |\bar{z}|$ when $|\bar{z}| \neq 0$, (14) can proved by choosing c and d as follows:

$$\begin{cases} (c, d) = \left(-\frac{h}{2}, \frac{2h}{3}\right) & \text{if } \bar{z} = 0, \\ (c, d) = (m_1 \bar{z}, m_2 \bar{z}) & \text{if } \bar{z} > 0, \\ (c, d) = (m_3 \bar{z}, m_4 \bar{z}) & \text{if } \bar{z} < 0. \end{cases}$$

Here m_i $(i = 1, 2, 3, 4)$ satisfy

$$\begin{cases} 1 - \frac{h}{\bar{z}} < m_1 < 1, \\ 1 + \frac{h}{3\bar{z}} < m_2 < 1 + \frac{h}{\bar{z}}, \\ 1 < m_3 < 1 - \frac{h}{\bar{z}}, \\ 1 + \frac{h}{\bar{z}} < m_4 < 1 + \frac{h}{3\bar{z}}. \end{cases}$$

Now let

$$\bar{F}_{jk} = \Big\{ t \in (\tau, t_1]\colon\ t \text{ is not a Lebesgue point of } \sup_{x \in B_{jk}} u(x,t) \Big\}$$

and set $\bar{F} = \bigcup \bar{F}_{jk}$. Then $\operatorname{meas}(\bar{F}) = 0$ and if $\bar{t} \notin \bar{F}$,

$$\lim_{t \to \bar{t}+} \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} \sup_{x \in B_{jk}} u(x,s)\, ds = \sup_{x \in B_{jk}} u(x, \bar{t})$$

and it is easy to see that we only need to prove that $(5)_1$ holds at $\bar{t} \notin A \cup D \cup E \cup \bar{F}$.

Since $y(t)$ and $z(t)$ are continuous functions, one can thus find $\Delta t > 0$ sufficiently small such that for $|t - \bar{t}| < \Delta t$,

$$y(t) < c, \qquad \bar{z} - h < c < z(t) < d < \bar{z} + h$$

and

$$4z(t) - c < 3d. \tag{15}$$

Thus if $|t - \bar{t}| < \Delta t$, we have $y(t) < c < \frac{2c + z(t)}{3} < z(t)$ and consequently $\rho(\cdot, t) = 0$ a.e. on $(\frac{2c+z(t)}{3}, z(t))$. Then we have from (13) of Lemma 1.2 that $\rho(\cdot, \bar{t}) = 0$ a.e. on

$$\left(\frac{2c + z(t)}{3} - \int_{\bar{t}}^{t} \inf_{x \in (c, \frac{4z(t)-c}{3})} u(x,s)\, ds,\ z(t) - \int_{\bar{t}}^{t} \sup_{x \in (c, \frac{4z(t)-c}{3})} u(x,s)\, ds \right),$$

and then we can get from the above result and (15) that

$$\bar{z} \geqslant z(t) - \int_{\bar{t}}^{t} \sup_{x \in (c, \frac{4z(t)-c}{3})} u(x,s)\, ds \geqslant z(t) - \int_{\bar{t}}^{t} \sup_{x \in B_{jk}} u(x,s)\, ds.$$

From which and (6), we have

$$\bar{u} + \varepsilon \leqslant \frac{1}{t - \bar{t}} \int_{\bar{t}}^{t} \sup_{x \in B_{jk}} u(x,s)\, ds.$$

Thus

$$\bar{u} + \varepsilon \leqslant \sup_{x \in B_{jk}} u(x, \bar{t})$$

which contradicts (7) since $B_{jk} \subset [\bar{z} - h, \bar{z} + h]$. The proof of $(5)_1$ is completed.

2. The proof of Lemma 1.2

This section is devoted to proving Lemma 1.2. The proof follows essentially the same way as in [1] and the main trick is to choose the test function $\phi^{\varepsilon\delta}$ with new coefficient $w^{\varepsilon\delta}$ and new initial data ψ^{δ} suitably.

First as in [1], we can get that $t_0 < t_1 < t_2$. Now suppose $t > t_1$; the proof for $t < t_1$ is similar and will be omitted. Fix $\delta > 0$ satisfying $\delta < \frac{b-a}{4}$, and for small $\varepsilon > 0$, let u^ε denote the usual spatial regularization of u. Then for almost all $t \in [t_1, T)$,

$$\left\|u^\varepsilon(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} \leqslant \left\|u(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)}.$$

For simplicity in notation, let

$$\begin{cases} \inf u^\varepsilon = \inf_{x\in(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} u^\varepsilon(\cdot,t), \\ \sup u^\varepsilon = \sup_{x\in(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} u^\varepsilon(\cdot,t), \\ \|u\|_\infty = \|u(\cdot,t)\|_{L^\infty(\frac{3a-b}{2},\,\frac{3b-a}{2})}, \end{cases}$$

one can easily get

$$\begin{aligned} -\infty < -\|u\|_\infty \leqslant -\left\|u^\varepsilon(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} \leqslant \inf u^\varepsilon \leqslant \sup u^\varepsilon \\ \leqslant \left\|u^\varepsilon(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} \leqslant \|u\|_\infty < \infty. \end{aligned}$$

Now define the smooth function $w^{\varepsilon\delta}$ by

$$w^{\varepsilon\delta}(x,t) = \begin{cases} \sup u^\varepsilon & \text{if } x < \frac{a+b}{2} - \delta, \\ \inf u^\varepsilon & \text{if } x > \frac{a+b}{2} + \delta, \end{cases}$$

and $w^{\varepsilon\delta}$ is decreasing on $(\frac{a+b}{2} - \delta, \frac{a+b}{2} + \delta)$.

Next, define the smooth function $\psi^\delta(x)$ by

$$\psi^\delta(x) = \begin{cases} 0 & \text{if } x < a+\delta, \\ 1 & \text{if } a+2\delta \leqslant x \leqslant b-2\delta, \\ 0 & \text{if } x > b-\delta, \end{cases}$$

and ψ^δ is increasing on the interval $(a+\delta, a+2\delta)$, and decreasing on $(b-2\delta, b-\delta)$.

Now let $\phi^{\varepsilon\delta}$ be the solution to the problem

$$\begin{cases} \phi_t + w^{\varepsilon\delta}\phi_x = 0, \quad t > t_1, \\ \phi(\cdot,t_1) = \psi^\delta. \end{cases} \tag{16}$$

By the characteristic method, it is easy to check that $\phi^{\varepsilon\delta}(x,t)$ satisfies

$$\begin{cases} \phi^{\varepsilon\delta}(x,t) \equiv 0 & \text{if } x \leqslant x_1(t) \equiv x_1(t; a+\delta, t_1), \\ \phi_x^{\varepsilon\delta}(x,t) > 0 & \text{if } x_1(t) < x < x_2(t) \equiv x_2(t; a+2\delta, t_1), \\ \phi^{\varepsilon\delta}(x,t) \equiv 1 & \text{if } x_2(t) \leqslant x \leqslant x_3(t) \equiv x_3(t; b-2\delta, t_1), \\ \phi_x^{\varepsilon\delta}(x,t) < 0 & \text{if } x_3(t) < x < x_4(t) \equiv x_4(t; b-\delta, t_1), \\ \phi^{\varepsilon\delta}(x,t) \equiv 0 & \text{if } x \geqslant x_4(t). \end{cases} \tag{17}$$

Here $x = x_i(t)$ $(i = 1,2,3,4)$ are characteristics passing through $(a+\delta, t_1)$, $(a+2\delta, t_1)$, $(b-2\delta, t_1)$, and $(b-\delta, t_1)$, respectively. That is, $\phi^{\varepsilon\delta}$ is a smooth, compactly supported function, and can thus serve as a test function for the (weak) formulation of a solution of (1.1), (1.2) in [1].

In particular, from (1.5) in [1], since for each $t \geqslant t_1$, $\operatorname{supp}\phi^{\varepsilon,\delta}(x,t) = [x_1(t), x_4(t)]$, we have

$$\int_{x_1(t)}^{x_4(t)} (\rho\phi^{\varepsilon\delta})(t,x)\,dx\bigg|_{t_1}^{t} = \int_{t_1}^{t}\int_{x_1(\tau)}^{x_4(\tau)} \rho\big(\phi_t^{\varepsilon\delta} + u\phi_x^{\varepsilon\delta}\big)\,dx\,d\tau$$
$$= \int_{t_1}^{t}\int_{x_1(\tau)}^{x_4(\tau)} \rho(u - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta}\,dx\,d\tau,$$

so that, since $x_1(t_1) = a+\delta$, $x_4(t_1) = b-\delta$, and $\rho(x,t_1) = 0$ a.e. for $x \in [a,b]$, we have

$$\int_{x_1(t)}^{x_4(t)} (\rho\phi^{\varepsilon\delta})(x,t)\,dx = \int_{t_1}^{t}\int_{x_1(\tau)}^{x_4(\tau)} \rho(u^{\varepsilon} - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta}\,dx\,d\tau + \int_{t_1}^{t}\int_{x_1(\tau)}^{x_4(\tau)} \rho(u - u^{\varepsilon})\phi_x^{\varepsilon\delta}\,dx\,d\tau. \tag{18}$$

Now define

$$T^{\varepsilon\delta} = \sup\Big\{t \in [t_1, T] \mid \text{The characteristics } x = x_2(t) \text{ and } x = x_3(t) \text{ stay } \delta \text{ units away from } \tfrac{a+b}{2} \text{ on } [t_1, t]\Big\}$$

as in [1], we can estimate $T^{\varepsilon\delta}$ from below as follows: Noticing that the characteristics of (16) are given by $\dot{x} = w^{\varepsilon\delta}$, we have along the characteristics $x = x_2(t; a+2\delta, t_1)$ that

$$\left(\frac{a+b}{2} - \delta\right) - (a+2\delta) = \int_{t_1}^{T^{\varepsilon\delta}} w^{\varepsilon\delta}\,d\tau \leqslant \int_{t_1}^{T^{\varepsilon\delta}} \sup u^{\varepsilon}\,d\tau \leqslant \int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty}\,d\tau,$$

while along the characteristics $x = x_3(t; b-2\delta, t_1)$,

$$(b-2\delta) - \left(\frac{a+b}{2} + \delta\right) = -\int_{t_1}^{T^{\varepsilon\delta}} w^{\varepsilon\delta}\,d\tau \leqslant -\int_{t_1}^{T^{\varepsilon\delta}} \inf u^{\varepsilon}\,d\tau \leqslant \int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty}\,d\tau.$$

Hence

$$\int_{t_1}^{T^{\varepsilon\delta}} \|u\|_{\infty}\,d\tau \geqslant \frac{b-a}{2} - 3\delta. \tag{19}$$

Therefore if T^{δ} is defined by

$$T^{\delta} = \sup\left\{t \in [t_1, T]\colon \int_{t_1}^{t} \|u\|_{\infty}\,d\tau < \frac{b-a}{2} - 3\delta\right\}, \tag{20}$$

then

$$T^{\varepsilon\delta} \geqslant T^{\delta}. \tag{21}$$

Thus if $t \in [t_1, T^{\delta}]$, then $t \in [t_1, T^{\varepsilon\delta}]$, so from (17), if $\phi_x^{\varepsilon\delta}(x,t) < 0$, then $x > \frac{a+b}{2} + \delta$, and $w^{\varepsilon\delta}(x,t) = \inf u^{\varepsilon}$.

Noticing that for $t \in [t_1, T^{\delta}]$,

$$\begin{aligned} x_1(t) &= a + \delta + \int_{t_1}^{t} w^{\varepsilon,\delta}\big(x_1(\tau),\tau\big)\,d\tau \geqslant a + \delta - \int_{t_1}^{t} \|u\|_{\infty}(\tau)\,d\tau \\ &\geqslant a + \delta - \left(\frac{b-a}{2} - 3\delta\right) = \frac{3a-b}{2} + 4\delta \end{aligned}$$

and

$$\begin{aligned} x_4(t) &= b - \delta + \int_{t_1}^{t} w^{\varepsilon,\delta}\big(x_4(\tau),\tau\big)\,d\tau \leqslant b - \delta + \int_{t_1}^{t} \|u\|_{\infty}(\tau)\,d\tau \\ &\leqslant b - \delta + \left(\frac{b-a}{2} - 3\delta\right) = \frac{3b-a}{2} - 4\delta, \end{aligned}$$

we have (17) and (18) that

$$\int_{t_1}^{t} \int_{x_1(\tau)}^{x_4(\tau)} \rho(u^{\varepsilon} - w^{\varepsilon\delta})\phi_x^{\varepsilon\delta}\,dx\,d\tau \leqslant 0. \tag{22}$$

Next, we claim that

$$\lim_{\varepsilon\to 0+} \int_{t_1}^{t} \int_{x_1(\tau)}^{x_4(\tau)} \rho(u - u^{\varepsilon})\phi_x^{\varepsilon\delta}\,dx\,d\tau = 0. \tag{23}$$

Assume this for the moment, we have from (18), (22), and (23) that

$$\overline{\lim_{\varepsilon\to 0+}} \int_{x_1(t)}^{x_4(t)} (\rho\phi^{\varepsilon\delta})(x,t)\,dx \leqslant 0, \quad t \in [t_1, T^{\delta}]. \tag{24}$$

Recall that the support of $\phi^{\varepsilon\delta}$ is the region bounded by the characteristics $x = x_1(t; a + \delta, t_1)$ and $x = x_4(t; b - \delta, t_1)$. Furthermore, due to

$$\begin{cases} \frac{dx_1(t)}{dt} = w^{\varepsilon,\delta}(x_1(t),t), \\ x_1(t_1) = a + \delta, \end{cases}$$

we have for $t_1 \leqslant t \leqslant T^{\delta}$ that

$$\begin{aligned} x_1(t) &= a + \delta + \int_{t_1}^{t} w^{\varepsilon,\delta}(x_1(\tau),\tau)\,d\tau \leqslant a + \delta + \int_{t_1}^{t} \|u\|_{\infty}(\tau)\,d\tau \\ &\leqslant a + \delta + \frac{b-a}{2} - 3\delta < \frac{a+b}{2} - \delta. \end{aligned}$$

Thus from the definition of $w^{\varepsilon,\delta}(t,x)$, we have for $t_1 \leqslant t \leqslant T^\delta$ that

$$w^{\varepsilon,\delta}\big(x_1(t),t\big) = \sup u^\varepsilon.$$

Hence for $t_1 \leqslant t \leqslant T^\delta$,

$$x_1(t) = a + \delta + \int_{t_1}^{t} \sup u^\varepsilon \, d\tau.$$

Similarly

$$x_4(t) = b - \delta + \int_{t_1}^{t} \inf u^\varepsilon \, d\tau.$$

Consequently the interval

$$I_\delta^\varepsilon = \left(a + \delta + \int_{t_1}^{t} \sup u^\varepsilon \, d\tau,\ b - \delta + \int_{t_1}^{t} \inf u^\varepsilon \, d\tau\right) \tag{25}$$

is contained in the support of $\phi^{\varepsilon\delta}(\cdot,t)$, then from (24) and the fact

$$\lim_{\varepsilon\to 0+} I_\delta^\varepsilon = I_\delta \equiv \left(a + \delta + \int_{t_1}^{t} \sup_{x\in(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} u \, d\tau,\right.$$
$$\left. b - \delta + \int_{t_1}^{t} \inf_{x\in(\frac{3a-b}{2}+4\delta,\,\frac{3b-a}{2}-4\delta)} u \, d\tau\right),$$

we can deduce for all $t \in [t_1, T^\delta]$ that

$$\rho(\cdot,t) = 0 \quad \text{a.e. on } I_\delta. \tag{26}$$

If now $t < t_2$, then

$$\int_{t_1}^{t} \|u\|_\infty \, d\tau < \frac{b-a}{2},$$

and thus there is a $\delta_0 > 0$ such that if $\delta < \delta_0$, then

$$\int_{t_1}^{t} \|u\|_\infty \, d\tau < \frac{b-a}{2} - 4\delta.$$

For such δ, (20) implies that $t \leqslant T^\delta$. Thus for such t and δ, $\rho(\cdot,t) = 0$ a.e. on I_δ. Taking a sequence $\delta_i \to 0+$, we get that $\rho(\cdot,t) = 0$ a.e. on the interval

$$\left(a + \int_{t_1}^{t} \sup_{x\in(\frac{3a-b}{2},\,\frac{3b-a}{2})} u(x,s)\, ds,\ b + \int_{t_1}^{t} \inf_{x\in(\frac{3a-b}{2},\,\frac{3b-a}{2})} u(x,s)\, ds\right)$$

for all $t \in [t_1, t_2]$, and this completes the proof of Lemma 1.2 when $t \geqslant t_1$.

It remains to prove (23). To this end, we first differentiate (16) with respect to x to obtain

$$\phi_{xt}^{\varepsilon\delta} + w^{\varepsilon\delta}\phi_{xx}^{\varepsilon\delta} = -w_x^{\varepsilon\delta}\phi_x^{\varepsilon\delta},$$

so that along the characteristics $x = x(t) \equiv x(t; x(t_1), t_1)$,

$$\phi_x^{\varepsilon\delta}\big(x(t), t\big) = \psi_x^{\delta}\big(x(t_1)\big) \exp\left(-\int_{t_1}^{t} w_x^{\varepsilon\delta}\big(x(s), s\big)\, ds\right). \tag{27}$$

Since

$$\left|w_x^{\varepsilon\delta}(\cdot, s)\right| \leqslant C(\delta)|\sup u^{\varepsilon} - \inf u^{\varepsilon}| \leqslant 2C(\delta)\|u\|_{\infty},$$

and thus from (27) we have there is a constant $C'(\delta)$ depending only on δ such that

$$\left\|\phi_x^{\varepsilon\delta}\right\|_{\infty} \leqslant C'(\delta).$$

Hence

$$\begin{aligned}
\left|\int_{t_1}^{t}\int_{x_1(\tau)}^{x_4(\tau)} \rho(u-u^{\varepsilon})\phi_x^{\varepsilon\delta}\, dx\, d\tau\right| &\leqslant C'(\delta)\int_{t_1}^{t} \left\|\rho(u-u^{\varepsilon})\right\|_{L^1(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)}\, d\tau \\
&\leqslant C'(\delta)\int_{t_1}^{T} \left\|u(\cdot,\tau) - u^{\varepsilon}(\cdot,\tau)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)} \\
&\quad \times \left\|\rho(\cdot,\tau)\right\|_{L^1(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)}\, d\tau.
\end{aligned} \tag{28}$$

But from hypothesis (A_4) of [1], we have that for almost all $t \in [t_1, T]$, $u(\cdot,t) \in H_{\mathrm{loc}}^1$ and from (1.7) of [1], $\|\rho(\cdot,t)\|_{L^1(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)}$ is bounded; thus for each fixed t, the integrand on the right-hand side of (28) tends to zero as $\varepsilon \to 0+$. Since

$$\begin{aligned}
&\left\|u(\cdot,t) - u^{\varepsilon}(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)} \left\|\rho(\cdot,t)\right\|_{L^1(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)} \\
&\quad \leqslant C(a,b)\left\|u(\cdot,t)\right\|_{L^\infty(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)}
\end{aligned}$$

and from Lemma 2.1 of [1], we know that $\|u(\cdot,t)\|_{L^\infty(\frac{3a-b}{2}+4\delta,\, \frac{3b-a}{2}-4\delta)}$ is integrable, the Lebesgue dominated convergence theorem applies to the right-hand side of (28) and shows that (23) holds. This completes the proof of Lemma 1.2. □

Acknowledgments

The authors are grateful to the referee for the valuable comments and suggestions which lead to the revision of the paper. This research was supported by a grant from the Chinese Academy of Sciences entitled "Yin Jin Guo Wai Jie Chu Ren Cai Ji Jin," the National Key Program for Basic Research of China under contract 2002CCA03700, and the National Natural Science Foundation of China under contracts 10329101 and 10431060, respectively.

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